Algebraic Solution to the Single Frequency Conjugate Match Impedance Problem

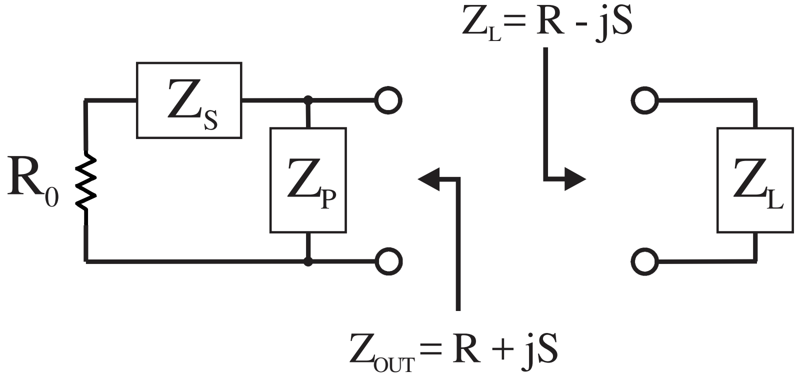
Thomas Mathews
Mathews Engineering

Problem: The traditional method to solve for a single frequency vector impedance match is to use the Smith Chart to find appropriate parallel and series matching components. Although this method is time tested, engineers often make mistakes and, if not used regularly, the Smith Chart can be very confusing. With the advent of powerful programs like Microsoft Excel, Mathcad, and Matlab it is no longer necessary to rely on the venerable, but obsolete Smith Chart.

Solution: Although I suspect it exists somewhere I was never able to find a closed-form solution to the single frequency conjugate match problem. A while ago I sat down and, after many hours of algebra, came up with these equations that will always find the exact solution to this matching problem. Admittedly these equations are hard to evaluate but are easily coded into programs like Excel or Mathcad.

For all equations: *R, R_0, and S are real numbers. (Z_S and Z_P will be purely imaginary)*
Because of the ± sign, each case has two valid solutions, one low-pass and one high-pass

Case 1: $R > R_0$

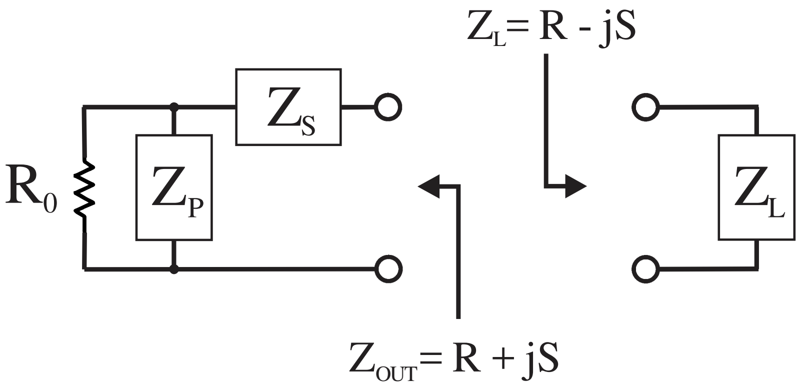


$$Z_P = \frac{-2jSR_0 \pm \sqrt{-4S^2R_0^{\,2} - 4R_0(R-R_0)(R^2+S^2)}}{2(R-R_0)}$$

$$Z_S = \frac{RR_0 + jSZ_P}{Z_P - jS}$$

Rev. A

Case 2: R < R_0



$$Z_P = \pm j\sqrt{\frac{RR_0^{\ 2}}{R_0 - R}}$$

$$Z_S = \frac{RR_0}{Z_P} + jS$$

Conversion of Zs and Zp to usable component values:

Convert positive imaginary values of Z to inductors (the j's will cancel):

$$L = \frac{Z}{j \cdot 2\pi f} \qquad \text{[Henrys]}$$

Convert negative imaginary values of Z to capacitors (the j's and negatives will cancel):

$$C = \frac{1}{j \cdot 2\pi f Z} \qquad \text{[Farads]}$$

Mathcad example (Case 1, R > R_0):

$R_0 := 50$

$R := 75$

$S := 10$

Case 1: R > R0, first solution

$$Z_{P1} := \frac{-2 \cdot j \cdot S \cdot R_0 + \sqrt{-4 \cdot S^2 \cdot R_0{}^2 - 4 \cdot R_0 \cdot (R - R_0) \cdot (R^2 + S^2)}}{2 \cdot (R - R_0)} \qquad Z_{P1} = 88.858i$$

$$Z_{S1} := \frac{R \cdot R_0 + j \cdot S \cdot Z_{P1}}{Z_{P1} - j \cdot S} \qquad Z_{S1} = -36.286i$$

Case 1: R > R0, second solution

$$Z_{P2} := \frac{-2 \cdot j \cdot S \cdot R_0 - \sqrt{-4 \cdot S^2 \cdot R_0{}^2 - 4 \cdot R_0 \cdot (R - R_0) \cdot (R^2 + S^2)}}{2 \cdot (R - R_0)} \qquad Z_{P2} = -128.858i$$

$$Z_{S2} := \frac{R \cdot R_0 + j \cdot S \cdot Z_{P2}}{Z_{P2} - j \cdot S} \qquad Z_{S2} = 36.286i$$

Mathcad Example (Case 2, R < R_0):

$R_0 := 300$

$R := 75$

$S := 25$

Case 2: R < R0, first solution

$$Z_{P1} := j \cdot \sqrt{\frac{R \cdot R_0{}^2}{R_0 - R}} \qquad Z_{P1} = 173.205i$$

$$Z_{S1} := \frac{R \cdot R_0}{Z_{P1}} + j \cdot S \qquad Z_{S1} = -104.904i$$

Case 2: R < R0, second solution

$$Z_{P2} := -j \cdot \sqrt{\frac{R \cdot R_0{}^2}{R_0 - R}} \qquad Z_{P2} = -173.205i$$

$$Z_{S2} := \frac{R \cdot R_0}{Z_{P2}} + j \cdot S \qquad Z_{S2} = 154.904i$$

Problems to watch out for:

1) Unrealistic component values: If you get very small values for capacitance or inductance (like fractions of a pico-Farad or a few nano-Henrys) then this is usually an indication that the match you are going for is too aggressive: ($R>>R_0$ or $R<<R_0$). Sometimes you can check the other solution (remember there is a ± in the equations). The other solution will sometimes lead to more realistic component values.

2) Limitations of component Q: Real world inductors and capacitors have finite quality factors. If you try to make a match that is too aggressive you may find that although theoretically correct, the actual match may be less satisfying because of lackluster component Q factors.

3) I get a complex result for Z_P or Z_S instead of a purely imaginary result?: This will happen if you are using the wrong equations for your situation. Make sure you are using **Case 1** equations for $R > R_0$ and **Case 2** equations for $R < R_0$

Author:
Thomas Mathews, PE, MSEE
Mathews Engineering
2931 N Webster Ave.
Indianapolis, IN 46219
www.mathews-engineering.com

Rev. A